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NAVAL COMBAT DAMAGE MODEL

J. G. Caldwell, et al

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13. ABSTRACT

This paper describes a model of naval combat which produces an assessment of damage to the combatants as a function of their respective force mixes. Both sides in the combat are permitted a number of force types, and the model works for a variety of combat scenarios. The model is sufficiently general to be of value in studies in which many strategy combinations and force types are to be examined.

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A Report on Work Performed Under
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NAVAL COMBAT DAMAGE MODEL

I. INTRODUCTION

During the past year, the primary objective of the contract under which this work was performed was to develop efficient methods for solving matrix games having multiple resource constraints (Reference 1). The objective was to develop methods which would work well for the variety of combat scenarios in which naval general purpose forces could be used. To satisfy this requirement, it was necessary to have a general framework or model in which the essential characteristics of a variety of force types could be specified, and with which the effectiveness of the forces used with and against other forces could readily be assessed. The model would have to be efficient enough to allow examination of the results of large numbers of strategy combinations. In other words, we needed a damage assessment model that was on the one hand quite general, and on the other hand, easy and inexpensive to use. Although several existing naval force engagement models were examined, no model was found to be sufficiently general or "automatic" enough for the present purposes. It was, therefore, considered necessary to develop a new damage model for the current research purpose. This paper describes this new damage model. It is felt that this model is general enough to be of residual value in other studies that may be undertaken dealing with naval general purpose forces.

In order to assess the value of naval general purpose forces in a combat situation, it is necessary to specify a means for assessing the damage to the combatants as a function of their respective force mixes. Specification of such a damage function is not an easy task, because of the varied roles that different force types play, and because of the interactions between force types. The damaging effects of sonobuoys, for example, is totally dependent on the amounts of other force types used in conjunction with the sonobuoys. Furthermore, force types can vary considerably with respect to several different essential characteristics, adding to the complexity of modeling the situation.

There are several basic properties that the damage function to be used in the current context must satisfy. These properties follow from the purpose for which the damage function is to be applied: it must enable a thorough test of the resource constrained matrix game-solving procedure. First, it must adequately reflect the fundamentally different roles of the force types: search, detection, engagement. Simple measures of "combat effectiveness" do not suffice - they mask the effects of interactions between force types. Second, the damage function must describe the effectiveness of a mix of forces against opposing forces. After a submarine is detected, for example, the probability that it is destroyed per unit of time changes as forces on the opposing side build up. Finally, the damage function must be readily computed, to hold the time and expense of testing the resource-constrained game problem to reasonable levels.

Several existing naval combat models were examined to determine whether or not they could be used as the damage function for the current effort. The models examined in detail were the APCAMP model (Reference 2), the submarine-carrier engagement model of CNA's AM&F study (Reference 3), and the models used by Dr. Paul Chaiken in his studies at Stanford Research Institute (Reference 4). As mentioned above, these models were considered not found to be suitable for the current purposes. They either did not consider interactions between enough force types, or simply did not model the engagement between forces in a manner appropriate to the purposes of the current research effort.

The primary reasons why the above models were not usable for the current study were the following:

1. APCAMP. This model had two drawbacks with regard to the current study. First, the way in which the forces were deployed had to be specified in detail by the user. It was not sufficient to specify deployment characteristics, and "let the model run". Second, the effectiveness of multiple numbers of forces on one side against multiple numbers of forces on the other side was not considered to be handled in a reasonable fashion. The interaction between forces of the same type, and between forces of different types, was essentially ignored.

2. CNA's AM&F Study. This study developed a model of the engagement between submarines and a carrier. The model treated search in a reasonable fashion, and modeled the effect of a single submarine against a carrier, also in a sound manner. (The differential equation approach to damage assessment that the study used was in fact adopted, in a more general context, for the current model.) The model treated the submarine attacks as independent, however, and did not allow for submarine communication to effect multiple submarine attacks. Furthermore, other force types (e.g., aircraft) were not considered.

3. Dr. Chaiken's Models. These models considered the effects of search, and modeled multiple-force number interactions through means of Lanchester equations. The models dealt with single force types in adequate detail, but did not consider the problem of communication or interactions between multiple force types. The Lanchester-type differential-equation approach to multiple force numbers was adopted for the current study, for the multiple-force-type situation.

Let us refer to the two combatants as "red" and "blue". This paper derives an expression for the expected numbers of red and blue forces destroyed, as a function of the numbers of red and blue forces deployed in the battle area. The situation modeled is one in which search, detection, and engagement all play important roles. A number of approximations and assumptions are made in order to obtain results that are analytically tractable, yet still reflect the essence of the problem.

II. MODEL SPECIFICATION

A. Introduction

In order for a damage function to be adequate for the present purposes, it must model both the search and detection, as well as the engagement, aspects of naval combat. One of the difficulties that arises in constructing such a general model stems from the fact that there are interactions between distinct engagements. Clearly, forces that join one engagement are not available to join another. If most of the forces were to be engaged at a point in time, it would be necessary to consider the implications of this effect. This situation is not, however, characteristic of combat in the campaign-length time frame with which we are concerned. To simplify the analysis, we shall therefore neglect the effect of this interaction. Furthermore, we shall impose a maximum time from the inception of an engagement during which forces may join that engagement.

The above assumptions are the principal restrictions to be imposed on the development of the model. There are, of course, a number of mathematical approximations made in constructing the model, but these will be described in the development of the model.

B. Qualitative Description of the Model

Before presenting the analytical development of the combat model, we shall give a verbal description of the salient model features, through means of an example. Suppose that each side possesses five force types: search aircraft, submarines, destroyers, sonobuoys, and mines (or mine fields). At any given point in time, the aircraft, submarines, and destroyers are searching for enemy submarines. Each force type has a speed and "search width," relative to each enemy force type. If an enemy force element of a particular type passes within the associated search width, he is detected. Upon detection, the detector informs all forces on his side of the detection. These other forces, with specified probabilities, may or may not choose to close on the detectee. The detector, with specified probability, may or may not choose to engage

the detectee. Thus a search aircraft that detects a submarine may choose not to engage the submarine, and rely on destroyers or other submarines to attack the detected submarine. Certain force types (e.g., other search aircraft) may, of course, choose not to engage the detected submarine. If a mine or mine field "detects" a submarine, it "engages" it with certainty, and has a specified probability of instantaneous kill. As other forces (destroyers, submarines) engage the detected submarine, the kill rate against the submarine changes, as does the submarines disengagement rate. After the maximum time mentioned earlier has expired, no new enemy forces may join the engagement. The engagement ends either when the submarine is destroyed or escapes, or all of the attacking forces disengage.

In the above situation, kill probabilities are of two types: (1) an initial discrete probability that the detectee is destroyed by a detector that chooses to engage the detectee; and (2) a continuous probability of kill per unit time (kill rate), that depends on the mix and numbers of forces attacking the detectee.

The rate at which forces are detected, or at which additional forces reach a detectee, depends on the densities of the force types over the battle area, and on their speeds. As we mentioned earlier, interactions between distinct engagements are not considered, and so the densities are not modified to reflect unavailable forces already engaged.

In order to most simply describe the value of a force type over a long period of time, it seems best to seek equilibrium, or "steady state" rates of kill of the various force types, assuming replacement of forces as they are destroyed. The principal reason for doing this is that it does not appear reasonable to allow long-time-period campaigns to "fight themselves out". It is the equilibrium kill rates, not the ultimate closed-system outcome, that best seems to describe the combat value of forces.

The analytical derivation of the combat damage model is presented in the Appendix.

III. COMPUTER PROGRAM

A computer program was written to implement the above analytical model. The following tables present some runs of the program. For the first case, each side had two weapon types, search aircraft and submarines, deployed in a 500,000 square nautical mile area. The aircraft could detect, but were not allowed to attack, the submarines. Submarines could both detect and attack enemy submarines, and always proceeded to engage detected enemy submarines, for up to 5 hours after the initial detection. The submarine's speed was 15 n.mi., the aircraft's speed was 150 n.mi. The search widths for both aircraft and submarines was 1 n.mi. The instantaneous kill probability of a submarine against a submarine was .5, the kill rate 1.0, and the disengagement rate .5. Kill rates were determined for all possible combinations of 0, 50, and 100 force elements of each type (due to symmetry, only half the cases were run). Table I shows the input data, and Table II shows the results of the run.

Table III shows the input data for a case involving 5 weapon types, and Table IV illustrates the results.

TABLE I.

Input Data, Case 1(Two Force Types)
(Data Given for Side 1 Only -- Side 2 is Identical.)

AREA = 500000.
MAX TIME TO JOIN ENGAGEMENTS = 5.00
SIDE 1 NO OF FORCE TYPES = 2
SIDE 2 NO OF FORCE TYPES = 2

FORCE TYPE NAMES FOR SIDE 1
SEARCH AIRCRAFT
HUNTER-KILLER SUBMARINE

FORCE TYPE NAMES FOR SIDE 2
SEARCH AIRCRAFT
HUNTER-KILLER SUBMARINE

DATA FOR SIDE 1
FORCE TYPE SPEEDS...
150.0000 15.0000 v_1 v_2
SEARCH WIDTHS...
DETECTEE TYPE 1
0.0000 0.0000 w_{11} w_{12}
DETECTEE TYPE 2
1.0000 1.0000 w_{21} w_{22}
PROBS THAT A NONDETECTOR HEADS TOWARD A DETECTEE...
DETECTEE TYPE 1
0.0000 0.0000 pab_{11} pab_{12}
DETECTEE TYPE 2
0.0000 1.0000 pab_{21} pab_{22}
PROBS THAT DETECTOR ENGAGES DETECTEE...
DETECTEE TYPE 1
0.0000 0.0000 pa_{11} pa_{12}
DETECTEE TYPE 2
0.0000 1.0000 pa_{21} pa_{22}
INSTANTANEOUS KILL PROB OF DETECTOR AG DETECTEE, GIVEN ENGAGEM
DETECTEE TYPE 1
0.0000 0.0000 pk_{11} pk_{12}
DETECTEE TYPE 2
0.0000 0.5000 pk_{21} pk_{22}
KILL RATE OF DETECTOR AG DETECTEE, IN EVENT OF NONIMMED KILL..
DETECTEE TYPE 1
0.0000 0.0000 α_{11} α_{12}
DETECTEE TYPE 2
0.0000 1.0000 α_{21} α_{22}
DISENGAGEMENT RATE OF DETECTEE FROM DETECTOR...
DETECTEE TYPE 1
0.0000 0.0000 β_{11} β_{12}
DETECTEE TYPE 2
0.0000 0.5000 β_{21} β_{22}
:

TABLE II.

Results, Case 1.

Side 1: Sub	Side 1: Plane	Side 2: Sub 0, Plane 0	Sub 0, Plane 50	Sub 0, Plane 100	Sub 50, Plane 0	Sub 50, Plane 50	Sub 50, Plane 100	Sub 100, Plane 0	Sub 100, Plane 50	Sub 100, Plane 100	
0	0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	
	50	0 0 0 0	0 .137 0 .137	0 .282* 0 .275	0 0 0 0	0 .640 0 .137	0 .785* 0 .275	0 0 0 0	0 1.142 0 .137	0 1.288* 0 .275	Kill rates for Side 1 Kill rates for Side 2
	100	0 0 0 0	0 .275 0 .282	0 .565 0 .565	0 0 0 0	0 1.280 0 .282	0 1.570 0 .565	0 0 0 0	0 2.285 0 .282	0 2.575 0 .565	
50	0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	
	50	0 0 0 0	0 .137 0 .640	0 .282 0 1.280	0 0 0 0	0 .640 0 .640	0 .785 0 1.280	0 0 0 0	0 1.142 0 .640	0 1.288 0 1.280	
	100	0 0 0 0	0 .275 0 .785	0 .565 0 1.570	0 0 0 0	0 1.280 0 .785	0 1.570 0 1.570	0 0 0 0	0 2.285 0 .785	0 2.575 0 1.570	
100	0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	
	50	0 0 0 0	0 .137 0 1.142	0 .282 0 2.285	0 0 0 0	0 .640 0 1.142	0 .785 0 2.285	0 0 0 0	0 1.142 0 1.142	0 1.288 0 2.285	
	100	0 0 0 0	0 .275 0 1.288	0 .565 0 2.575	0 0 0 0	0 1.288 0 1.280	0 1.570 0 2.575	0 0 0 0	0 2.285 0 1.288	0 2.575 0 2.575	

*Consistency check: .785 + (.715 - .281) = .785 + .503 = 1.288

TABLE III.

Input Data, Case 2 (Five Force Types)
(Data Given for Side 1 Only -- Side 2 is Identical.)

```
AREA =        500000.
MAX TIME TO JOIN ENGAGEMENTS =       5.00
SIDE 1 NO OF FORCE TYPES =    5
SIDE 2 NO OF FORCE TYPES =    5

FORCE TYPE NAMES FOR SIDE     1
SEARCH AIRCRAFT
HUNTER-KILLER SUBMARINE
DESTROYER
MINE FIELD
SONOBUOY

FORCE TYPE NAMES FOR SIDE     2
SEARCH AIRCRAFT
HUNTER-KILLER SUBMARINE
DESTROYER
MINE FIELD
SONOBUOY

DATA FOR SIDE    1
FORCE TYPE SPEEDS...
  150.0000   20.0000   20.0000    0.0000    0.0000
SEARCH WIDTHS...
DETECTEE TYPE     1
    0.0000    0.0000    0.0000    0.0000    0.0000
DETECTEE TYPE     2
    1.0000    1.0000    1.0000    1.0000    1.0000
DETECTEE TYPE     3
    0.0000    0.0000    0.0000    0.0000    0.0000
DETECTEE TYPE     4
    0.0000    0.0000    0.0000    0.0000    0.0000
DETECTEE TYPE     5
    0.0000    0.0000    0.0000    0.0000    0.0000
PROBS THAT A NONDETECTOR HEADS TOWARD A DETECTEE...
DETECTEE TYPE     1
    0.0000    0.0000    0.0000    0.0000    0.0000
DETECTEE TYPE     2
    0.0000    1.0000    1.0000    0.0000    0.0000
DETECTEE TYPE     3
    0.0000    0.0000    0.0000    0.0000    0.0000
DETECTEE TYPE     4
    0.0000    0.0000    0.0000    0.0000    0.0000
DETECTEE TYPE     5
    0.0000    0.0000    0.0000    0.0000    0.0000
```

TABLE III. (conc.)

PROBS THAT DETECTOR ENGAGES DETECTEE...

DETECTEE TYPE				
1				
0.0000	0.0000	0.0000	0.0000	0.0000
2				
0.0000	1.0000	1.0000	1.0000	0.0000
3				
0.0000	0.0000	0.0000	0.0000	0.0000
4				
0.0000	0.0000	0.0000	0.0000	0.0000
5				
0.0000	0.0000	0.0000	0.0000	0.0000

INSTANTANEOUS KILL PROB OF DETECTOR AG DETECTEE, GIVEN ENGAGEMENT...

DETECTEE TYPE				
1				
0.0000	0.0000	0.0000	0.0000	0.0000
2				
0.0000	0.5000	0.5000	0.2500	0.0000
3				
0.0000	0.0000	0.0000	0.0000	0.0000
4				
0.0000	0.2500	0.0000	0.0000	0.0000
5				
0.0000	0.0000	0.0000	0.0000	0.0000

KILL RATE OF DETECTOR AG DETECTEE, IN EVENT OF NONIMMED KILL...

DETECTEE TYPE				
1				
0.0000	0.0000	0.0000	0.0000	0.0000
2				
0.0000	1.0000	1.0000	0.0000	0.0000
3				
0.0000	0.0000	0.0000	0.0000	0.0000
4				
0.0000	0.0000	0.0000	0.0000	0.0000
5				
0.0000	0.0000	0.0000	0.0000	0.0000

DISENGAGEMENT RATE OF DETECTEE FROM DETECTOR...

DETECTEE TYPE				
1				
0.0000	0.0000	0.0000	0.0000	0.0000
2				
1.0000	1.0000	1.0000	1.0000	1.0000
3				
0.0000	0.0000	0.0000	0.0000	0.0000
4				
0.0000	0.0000	0.0000	0.0000	0.0000
5				
0.0000	0.0000	0.0000	0.0000	0.0000

TABLE IV.

Results, Case 2.

SIDE	FORCE LEVELS					CORRESPONDING KILL RATES				
	PLANE	SUB.	DEST.	MINE	SONO.	PLANE	SUB.	DEST.	MINE	SONO.
1	100	100	100	100	100	0	1.891	0	.1†	0
2	100	100	100	100	100	0	1.891*	0	.1	0
1	100	100	100	100	100	0	1.622	0	.1	0
2	50	100	100	100	100	0	1.891	0	.1	0
1	100	100	100	100	100	0	1.442	0	.05†	0
2	100	50	100	100	100	0	.945*	0	.1	0
1	100	100	100	100	100	0	1.556	0	.1	0
2	100	100	50	100	100	0	1.891	0	.1	0
1	100	100	100	100	100	0	1.814	0	.0	0
2	100	100	100	50	100	0	1.891	0	.05	0
1	100	100	100	100	100	0	1.855	0	.1	0
2	100	100	100	100	50	0	1.891	0	.1	0

Consistency Checks: All kill rates move in correct directions.

*Also, 1.891/2 = .945
†Alsc, .1/2 = .05

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APPENDIX

ANALYTICAL DEVELOPMENT OF DAMAGE MODEL

I. Engagement Model

In this section we shall develop a model of the course and outcome of an engagement, conditional on a detection of a specified red force type by a specified blue force type. The next section will combine this result with an expression for the expected numbers of detections of the various types per unit time, to yield an expression for the total expected numbers of forces destroyed per unit time, (i.e., for the kill rates).

The following notation will be used to define the variables in terms of which the model is constructed. The variables given below will represent the red forces. A bar above a symbol will be used to denote a corresponding quantity for blue. We shall refer to a particular member of force type i as an "element of type i," or simply as "an i." Let

pa_{ij} = probability that i attacks an enemy j, given that the i detected the j;

pab_{ij} = probability that a nondetctor i heads toward a detected enemy j;

pk_{ij} = probability that an i kills an enemy j immediately, given that the i detected and then engaged the j;

α_{ij} = kill rate of an i against an enemy j, given that the i is engaged against the j (and, of course, the j is still alive);

β_{ij} = disengagement rate for a j when engaged by an enemy i;

N_i = number of forces of type i in the battle area;

A = area of the battle area;

ρ_i = density of force type i = N_i/A;

t_m = maximum time for forces to join an engagement (measured from the time of detection);

w_{ij} = "search width" of an i for an enemy j (i.e., if an enemy j comes within distance w_{ij} of an i, then the i detects the j;

v_i = speed of an i.

We make the following assumptions about the effectiveness of the forces and the manner in which they are employed.

1. If an i detects an enemy j, only forces on the i's side may enter the resultant engagement involving the j (we refer to the arrival of new forces as "buildup");

2. The kill rate and disengagement rate of an i against an enemy j are unaffected by the presence of other forces on either side; kill rates and disengagement rates are additive;

3. If an i detects an enemy j, either the i or the j may be killed, but no additional casualties may result.

If an i detects an enemy j, the following events may occur:

1. The i does not attack the j;

2. The i attacks the j, in which case the following events may occur to the j:

   a. the i kills the j immediately, ending the engagement;

   b. the i kills the j in time, ending the engagement;

   c. the j is killed by buildup, ending the engagement;

   d. the j disengages, ending the engagement;

and the following events may occur to the i:

   e. the j kills the i immediately, ending the engagement;

   f. the j kills the i in time (but the j is still subject to kill by buildup).

Let us assume that it is a blue j that detects a red i. We have

$$\begin{aligned} pdef(i,j) &= P(\text{red i is killed} \mid \text{blue j detects red i}) \\ &= P(\text{i is killed} \mid \text{j doesn't attack})P(\text{j doesn't attack}) \\ &\quad + P(\text{i is killed} \mid \text{j attacks})P(\text{j attacks}) \\ &= p_1(i,j)(1 - \overline{pa}_{ji}) + p_2(i,j)\overline{pa}_{ij}, \end{aligned}$$

where

$$p_1(i,j) = P(i \text{ is killed} \mid j \text{ doesn't attack})$$

$$p_2(i,j) = P(i \text{ is killed} \mid j \text{ attacks}).$$

We shall now derive expressions for p_1 and p_2. Now

$$\begin{aligned} p_1(i,j) &= 1 - P(i \text{ survives buildup} \mid j \text{ doesn't attack}) \\ &= 1 - P(\text{disengagement event occurs before kill event occurs}) \\ &= 1 - \int_t P(\text{disengagement event at time } t \text{ and kill event after } t)\, dt \\ &= 1 - \int_t P(\text{disengagement event at } t)\, P(\text{kill event after } t)\, dt \end{aligned}$$

since we are assuming disengagement and kill "events" occur independently. Now

$$P(\text{kill event occurs after } t) = P(\text{survival to } t \mid \text{no disengagement by } t).$$

Let p_t denote the preceding probability. We have

$$p_{t+\Delta t} = p_t(1 - r_t \Delta t)$$

where r_t denotes the kill rate due to all arriving enemy forces. In time t after detection, this kill rate is

$$r_t = \begin{cases} \sum_k \bar{\alpha}_{ki}\, \pi\, (\bar{v}_k t)^2\, \bar{\rho}_k\, \overline{pab}_{ki} & \text{if } t \le t_m \\ \sum_k \bar{\alpha}_{ki}\, \pi\, (\bar{v}_k t_m)^2\, \bar{\rho}_k\, \overline{pab}_{ki} & \text{if } t > t_m \end{cases}$$

$$= \begin{cases} a_i t^2 & t \le t_m \\ a_i t_m^2 & t > t_m \end{cases}$$

where

$$a_i = \sum_k \bar{\alpha}_{ki} \pi \, (\bar{v}_k t)^2 \, \bar{\rho}_k \, \overline{pab}_{ki} \; .$$

Taking the limit as $\Delta t \to 0$ and solving the resultant differential equation, we obtain

$$p_t = \begin{cases} e^{- a_i t^3/3} & t \le t_m \\ e^{- a_i t_m^2 t \, + \, 2 a_i t_m^3/3} & t > t_m \end{cases}$$

as the probability that the kill event occurs after time t (i.e., that the i survives to time t, given that he has not yet disengaged). Similarly, if we now redefine r_t to be the disengagement rate due to the detector and all arriving enemy forces, we have

$$r_t = \begin{cases} \sum_k \bar{\beta}_{ki} \pi (\bar{v}_k t)^2 \, \bar{\rho}_k \, \overline{pab}_{ki} & t \le t_m \\ \sum \bar{\beta}_{ki} \pi (\bar{v}_k t_m)^2 \, \bar{\rho}_k \, \overline{pab}_{ki} & t > t_m \end{cases}$$

$$= \begin{cases} b_i t^2 & t \le t_m \\ b_i t_m^2 & t > t_m \end{cases}$$

where

$$b_i = \sum_k \bar{\beta}_{ki} \pi (\bar{v}_k t)^2 \, \bar{\rho}_k \, \overline{pab}_{ki} \; .$$

If we redefine p_t now to be the probability that the disengagement event has occurred by time t, we obtain

$$p_t = \begin{cases} e^{-(\bar{\beta}_{ji}t + b_i t^3/3)} & t \leq t_m \\ e^{-(\bar{\beta}_{ji} + b_i t_m^2)t + 2b_i t_m^3/3} & t > t_m \end{cases}$$

Hence, the probability that the disengagement occurs by time t is $1 - p_t$, and the probability element that the disengagement occurs at time t is

$$\begin{cases} \left(\bar{\beta}_{ji} + b_i t^2\right) e^{-(\bar{\beta}_{ji}t + b_i t^3/3)} dt & t \leq t_m \\ \left(\bar{\beta}_{ji} + b_i t_m^2\right) e^{-(\bar{\beta}_{ji} + b_i t_m^2)t + 2b_i t_m^3/3} dt, & t > t_m \ . \end{cases}$$

Hence we have

$$p_1(i,j) = 1 - \int_0^{t_m} (\bar{\beta}_{ji} + b_i t^2) e^{-(\bar{\beta}_{ji}t + b_i t^3/3)} e^{-a_i t^3/3} dt$$

$$+ \int_{t_m}^{\infty} (\bar{\beta}_{ji} + b_i t_m^2) e^{-(\bar{\beta}_{ji} + b_i t_m^2)t + 2b_i t_m^2/3}$$

$$\cdot e^{-a_i t_m^2 + 2a_i t_m^3/3} dt \ .$$

This integral is difficult to evaluate, and we shall hence seek an approximation to it. For very small t, kill is unlikely, and the disengagement rate is $\bar{\beta}_{ji}$. For all $t \leq t_m$, the ratio of the disengagement rate to the sum of the disengagement and kill rates is

$$\frac{\bar{\beta}_{ji} + b_i t^2}{\bar{\beta}_{ji} + (a_i + b_i)t^2} \ .$$

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(Note: the detector's kill rate $\overline{\alpha}_{ji}$ does not appear in the denominator, since we are considering the case in which the detector does not attack.) For $t \geq t_m$, the ratio is

$$\frac{\overline{\beta}_{ji} + b_i t_m^2}{\overline{\beta}_{ji} + (a_i + b_i) t_m^2} .$$

For large t_m this quantity approaches $b_i/(a_i + b_i)$. For small t_m, this quantity approaches 1. For

$$t_{1j} = \sqrt{\frac{\overline{\beta}_{ji}}{a_i + b_i}}$$

this quantity is halfway between the two limits. If we approximate

$$\frac{\overline{\beta}_{ji} + b_i t^2}{\overline{\beta}_{ji} + (a_i + b_i) t^2}$$

by 1 for $t \leq t_{1j}$ and by

$$\frac{\overline{\beta}_{ji} + b_i t_m^2}{\overline{\beta}_{ji} + (a_i + b_i) t_m^2}$$

for $t > t_{1j}$, we have, for $t_m \geq t_{1j}$,

$$P(\text{i survives buildup} \mid \text{j doesn't attack}) = P(\text{disengagement event occurs by } t_{1j}) \cdot 1 + (1 - P(\text{disengagement event occurs by } t_{1j})) \cdot \frac{\overline{\beta}_{ji} + b_i t_m^2}{\overline{\beta}_{ji} + (a_i + b_i) t_m^2} ,$$

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and, for $t_m < t_{1j}$

P(i survives buildup | j doesn't attack) = P(disengagement event occurs by t_m)·1 + (1 - P(disengagement event occurs by t_m)).

$$\cdot \frac{\bar{\beta}_{ji} + b_i t_m^2}{\bar{\beta}_{ji} + (a_i + b_i) t_m^2} .$$

Now

$$P(\text{disengagement event occurs by } t) = \begin{cases} 1 - e^{-\bar{\beta}_{ji} t_{1j}} & \text{for } t = t_{1j} \\ 1 - e^{-\bar{\beta}_{ji} t_m} & \text{for } t = t_m . \end{cases}$$

Hence we have

$$P_1(i,j) = 1 - P(\text{i survives buildup} \mid \text{j doesn't attack})$$

$$= \begin{cases} e^{-\bar{\beta}_{ji} t_{1j}} \dfrac{a_i t_m^2}{\bar{\beta}_{ji} + (a_i + b_i) t_m^2} & t_{1j} \le t_m \\[2ex] e^{-\bar{\beta}_{ji} t_m} \dfrac{a_i t_m^2}{\bar{\beta}_{ji} + (a_i + b_i) t_m^2} & t_{1j} > t_m . \end{cases}$$

Having derived an expression for $p_1(i,j)$, we now proceed to derive an expression for $p_2(i,j)$. Now

$$
\begin{aligned}
p_2(i,j) &= P(\text{i is killed} \mid \text{j attacks}) \\
&= P(\text{i is killed immediately} \mid \text{j attacks}) \\
&\quad + P(\text{i is killed in time} \mid \text{j attacks and i is not killed} \\
&\qquad \text{immediately}).\ P(\text{i is not killed immediately} \mid \text{j attacks} \\
&= \overline{pk}_{ji} + P(\text{i is killed} \mid \text{j attacks and i is not killed} \\
&\qquad \text{immediately})(1 - \overline{pk}_{ji}) \\
&= \overline{pk}_{ji} + p_3(i,j)(1 - \overline{pk}_{ji}),
\end{aligned}
$$

say, where

$$p_3(i,j) = P(\text{i is killed} \mid \text{j attacks and i is not killed immediately}).$$

Let A denote the event

$$\left\{ \text{j attacks and i is not killed immediately} \right\}.$$

Then

$$
\begin{aligned}
p_3(i,j) &= P(\text{event "j kills i" occurs} \mid A) + P(\text{event "buildup kills i"} \\
&\qquad \text{occurs} \mid A) - P(\text{event "j kills i" occurs} \mid A) \\
&\qquad \cdot P(\text{event "buildup kills i" occurs}) \\
&= p_4(i,j) + p_1(i,j) - p_4(i,j)p_1(i,j),
\end{aligned}
$$

where

$$
\begin{aligned}
p_4(i,j) &= P(\text{event "j kills i" occurs} \mid A) \\
&= P(\text{event "j kills i" occurs before event "i kills j" occurs}) \\
&\quad \cdot P(\text{event "j kills i" occurs before event "i disengages j"} \\
&\qquad \text{occurs}) \\
&\quad \cdot P(\text{event "j kills i" occurs before event "j disengages i"} \\
&\qquad \text{occurs})
\end{aligned}
$$

$$= \frac{\overline{a_{ji}}}{\overline{a_{ji}} + \overline{a_{ij}}} \cdot \frac{\overline{a_{ji}}}{\overline{a_{ji}} + \overline{b_{ji}}} \cdot \frac{\overline{a_{ji}}}{\overline{a_{ji}} + b_{ij}} .$$

Hence, substituting p_4 into p_3, and p_3 into p_2, we obtain the expres for $p_2(i,j)$. Having expressions for both $p_1(i,j)$ and $p_2(i,j)$, we hence have evaluated

$$\text{pdef}(i,j) = P(\text{red } i \text{ is killed} | \text{blue } j \text{ detects red } i).$$

Now if a blue j detects a red i, the probability that the j is destroyed is

$$\begin{aligned}
\overline{\text{poff}}(j,i) &= P(\text{blue } j \text{ is killed} | \text{blue } j \text{ detects red } i) \\
&= P(j \text{ attacks} | j \text{ detects } i) \\
&\quad (P(i \text{ kills } j \text{ immediately} | j \text{ attacks}) \\
&+ P(j \text{ kills } i \text{ in time} | j \text{ attacks and } i \text{ is not killed imm} \\
&\cdot P(i \text{ does not kill } j \text{ immediately} | j \text{ attacks})).
\end{aligned}$$

II. Total Kill Rates

We have now determined the kill probabilities, conditional on detection. In order to determine the kill rates for the various force types, it remains simply to determine the detection rates. We have

$$\begin{aligned}
d(i,j) &= \text{rate at which } i \text{ detects } j \\
&= N_i N_j (v_i^2 + \overline{v}_j^2)^{1/2} w_{ij} / A.
\end{aligned}$$

Hence

$$\begin{aligned}
rk(i) &= \text{kill rate for red force type } i \\
&= \sum_j \text{poff}(i,j) d(i,j) + \text{pdef}(i,j) \overline{d}(j,i)
\end{aligned}$$

and

$$\overline{rk}(j) = \text{kill rate for blue force type } j$$
$$= \sum_i \overline{poff}(j,i)\overline{d}(j,i) + \overline{pdef}(j,i)d(i,j).$$